

The Two-Dimensional $O(2)$ Model on a Random Planar Lattice at Strong Coupling

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Abstract

The large spacing phase of the infinite random matrix chain, which represents the strongly coupled two-dimensional $O(2)$ model on a random planar lattice, is explored. A class of solutions valid for large lattice spacings is constructed. It is proved that these solutions exhibit the critical exponents characteristic of pure two-dimensional gravity. The character expansion for the chain model is developed and an order parameter governing the Kosterlitz–Thouless phase transition is identified.

1. Introduction

String theories with the matter central charge one are of special interest. They are positioned in between the exactly solved $c < 1$ minimal models interacting with quantum gravity [1] and the yet mysterious $c > 1$ noncritical string theories [2][3].

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The $c = 1$ string where the target space is an infinite one-dimensional line can also be analyzed exactly [4]. However, new problems arise once the target space is compactified. In that case the theory has been solved only for sufficiently large values of the compactification radius $R > R_{\text{cr}}$. At $R = R_{\text{cr}}$ the $c = 1$ model undergoes a phase transition, and there is evidence that for smaller R it behaves like a theory of pure two-dimensional gravity, without any matter at all [5][6].

The transition at $R = R_{\text{cr}}$ is quite similar in nature to the famous Kosterlitz–Thouless phase transition in the two-dimensional $O(2)$ model [7]. Naively, these transitions are not automatically present in continuum theory and appear only if the worldsheet is discretized. Both of them are induced by topologically nontrivial field configurations—vortices—where the compactified string, or a two-component unit vector in the $O(2)$ model winds around a unit circle as we follow the boundary of an elementary worldsheet plaquette. The dynamical role of vortices in the theory depends on how the vortex energy and entropy compare to each other. As a result, while the effect of vortices is negligible for $R > R_{\text{cr}}$ (or in the weakly coupled phase of the $O(2)$ model) the vortices actually dominate the dynamics at $R < R_{\text{cr}}$ (respectively, in the strong coupling phase of the $O(2)$ model.) Furthermore, the well known $R \leftrightarrow \alpha'/R$ duality of one-dimensional compactified string theory holds only if the vortices are completely ignored. It is possible to show [5] that the contributions to the free energy coming from the vortexless sector of the $c = 1$ theory do respect the T -duality, whereas the vortex contributions manifestly break it.

To understand the dynamics of vortices interacting with quantum gravity is a long-standing problem [5][6]. Technically, this problem can be formulated in a very natural way using the language of random matrix models. Indeed, there the two-dimensional worldsheet is discretized by construction. Thus any matrix model describing the compactified one-dimensional string theory will contain vortices from the very beginning. The difficulty is, all such models involve an infinite number of interacting matrices and, because of that, have not yet been solved.

In this paper we shall explore the simplest among these models, a one-dimensional infinite random matrix chain defined by the partition function [5][8][9][10]

$$\mathcal{Z} = \int \prod_{n=-\infty}^{+\infty} dM_n \exp \left\{ -N \text{Tr} \sum_{n=-\infty}^{+\infty} \left[\frac{(M_{n+1} - M_n)^2}{2\epsilon} + \epsilon V(M_n) \right] \right\}. \quad (1.1)$$

Here all M_n are $N \times N$ Hermitian matrices while $V(M)$ is a polynomial potential such as, for example, $m^2 M^2/2 + \tilde{g} M^3/3$ or $m^2 M^2/2 + \tilde{g} M^4/4$.

It is possible to show [5] that the large N limit of the chain model does indeed describe the one-dimensional bosonic string theory compactified on a circle of radius $R = 1/\epsilon$ or,

equivalently, the two-dimensional $O(2)$ nonlinear sigma model coupled to quantum gravity. That is to say, the leading term in the large N expansion of \mathcal{Z} summarizes correctly the vortex properties on topologically spherical random surfaces. At the same time the $1/N$ corrections to the chain partition function do not represent the higher genus compactified $c = 1$ amplitudes. Instead, they correspond to a different theory—a one-dimensional string with discretized target space. On a spherical worldsheet this theory is related to the compactified one-dimensional string by a duality transformation, and the string partition functions for the two coincide.

It has long been known [5] that the infinite matrix chain undergoes a Kosterlitz–Thouless phase transition at a certain $\epsilon = \epsilon_{\text{cr}}$. Throughout the region $\epsilon < \epsilon_{\text{cr}}$ where vortices can be neglected the critical indices of (1.1) do correspond to $c = 1$. On the other hand, very little, if anything, is known about the properties of the matrix chain for $\epsilon > \epsilon_{\text{cr}}$. Quite remarkably, it is this regime, totally dominated by vortices, that is the hardest to analyze by traditional means.

Certainly, in the limit of infinitely large ϵ the matrix chain partition function decouples into a product of independent one-matrix models. Therefore, one might conjecture that, at least for large ϵ , the critical indices of \mathcal{Z} are those of pure two-dimensional gravity. We shall see that such a conclusion is in fact true for all $\epsilon > \epsilon_{\text{cr}}$. As a consequence, the effects of dimensional reduction in string theories and field theories may in fact be much more similar than one usually believes. Consider, say, a field theory on a manifold with a compact dimension. When the compactification radius vanishes the compactified dimension effectively disappears, and we obtain a (perhaps modified) field theory in one less dimension. In string theory the picture is quite different. There because of the $R \leftrightarrow \alpha'/R$ duality the small and large compactification radii appear to be strictly equivalent, and taking R to zero does not reduce the effective dimension. However, once the nonperturbative effects—such as vortices—are included, the situation may change drastically and the parallels with field theory may get restored. Exactly that happens in our one-dimensional example where for $R < R_{\text{cr}}$ the compactified string theory behaves as if the one-dimensional target space circle had completely disappeared and the target space dimension was equal to zero.

Below we shall construct, for certain $V(M)$, an exact solution of the matrix chain valid at $\epsilon > \epsilon_{\text{cr}}$. Remarkably, the computational tools needed for that are quite interesting on their own. As we shall see, the large N asymptotics of the matrix chain model is related to a certain one-dimensional hydrodynamic system. The Kosterlitz–Thouless phase transition corresponds then to the formation of a shock-type configuration in the moving fluid.

To state this more precisely, let us first rescale all matrices $M_n = \sqrt{\epsilon}\mathcal{M}_n$ so that ϵ disappears from the kinetic term in the partition function,

$$\mathcal{Z} = \int \prod_{n=-\infty}^{\infty} d\mathcal{M}_n \exp\left\{N \operatorname{tr} \sum_{n=-\infty}^{\infty} [\mathcal{M}_n \mathcal{M}_{n+1} - U(\mathcal{M}_n)]\right\} \quad (1.2)$$

the new potential $U(\mathcal{M})$ being related to $V(M)$ by

$$U(\mathcal{M}) = \mathcal{M}^2 + \epsilon V(\sqrt{\epsilon}\mathcal{M}). \quad (1.3)$$

At $N \rightarrow \infty$ the eigenvalues $\lambda_{1,n} \dots \lambda_{N,n}$ of any given matrix \mathcal{M}_n in (1.2) condense to form a smooth distribution $\rho_n(\lambda)$. Actually, due to the translational invariance of the matrix chain this distribution is the same for any chain site, $\rho_n(\lambda) \equiv \rho(\lambda)$. It turns out that the density $\rho(\lambda)$ can be found as a solution to the following hydrodynamic problem [11][12]. Consider a one-dimensional droplet of compressible fluid with a special equation of state

$$P = -\frac{\pi^2}{3}\rho^3 \quad (1.4)$$

where P is the local pressure and ρ —the local fluid density. Imagine that at time $t = 0$ the density ρ equals (the unknown) $\rho(x)$ and the initial fluid velocity is

$$v(x) = \frac{1}{2}U'(x) - x. \quad (1.5)$$

Now, demand that after one unit of time, at $t = 1$, the density of the droplet returns to its initial value,

$$\rho(x, t = 1) = \rho(x, t = 0) = \rho(x). \quad (1.6)$$

This condition fixes $\rho(x)$ once $U(x)$ (and hence $v(x)$) is given, so that the problem of the matrix chain reduces to solving the hydrodynamic Euler equations.

As a consequence of (1.6), together with the time reversal symmetry of Euler equations, the initial and final fluid velocities are opposite, $v(x, t = 1) = -v(x, t = 0)$ and, furthermore, $v(x, t = 1/2) \equiv 0$. The density at $t = 1/2$, which we denote $\rho_{1/2}(x)$, shall play a very important role in our analysis. Its value at the origin $\zeta = \rho_{1/2}(0)$ provides an “order parameter” for the Kosterlitz–Thouless transition. That is to say, $\zeta \equiv 0$ for any $\epsilon < \epsilon_{\text{cr}}$ but $\zeta > 0$ everywhere in the phase of large ϵ .

Of course, the Kosterlitz–Thouless transition is not explicitly associated with the spontaneous breaking of any symmetry and using the term “order parameter” may appear purely formal. However, this phase transition does reflect a nonanalytic change in the group representation structure of (1.1) at large N . In section 4 we shall demonstrate that the integrand of \mathcal{Z} can be expanded as a series over characters of $SU(N)$. At infinite N only one representation in this series, with the Young tableau row lengths (n_1, \dots, n_N) all of order N , makes an important contribution. Such representations are usually characterized by the Young tableau density $\rho_l(h)$ defined as the density of points $h_i = (n_i - i)/N + 1$ in a small interval around h . Since the row lengths are all ordered, $\rho_l(h)$ can never be greater than one.

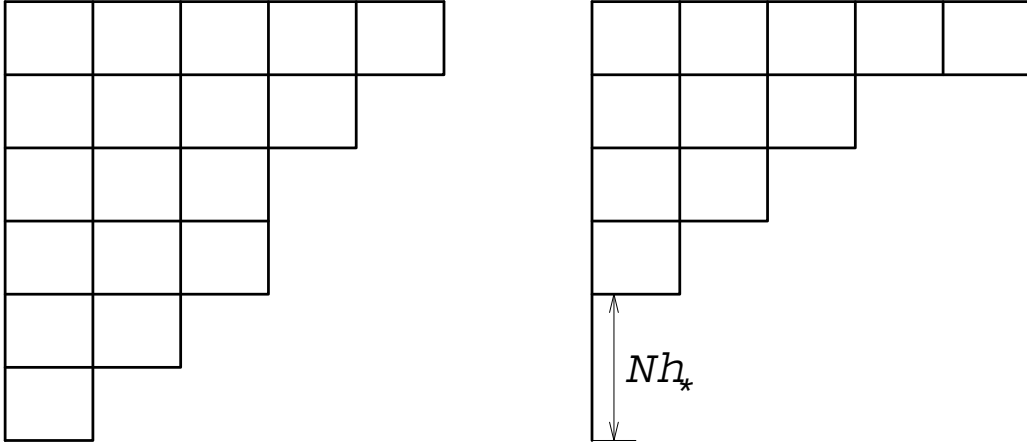


Fig. 1: The Young tableaux for the representations dominant at $\epsilon < 1$ (left) and $\epsilon > 1$ (right).

Whenever the Young tableau density saturates this upper bound, $\rho_l(h_*) = 1$, the corresponding Young tableau develops a gap. It turns out that such gap formation is just another aspect of the same Kosterlitz–Thouless phase transition at $\epsilon = \epsilon_{\text{cr}}$. Moreover, the “order parameter” $\zeta = \rho_{1/2}(0)$ happens to be very directly related to the gap size h_* ,

$$\zeta = \rho_{1/2}(0) = \frac{2}{\pi} \sqrt{h_*}. \quad (1.7)$$

Such a correspondence unifies, at least technically, the Kosterlitz–Thouless transition and the Douglas–Kazakov phase transitions known from two-dimensional QCD and the dually weighted graph models [13][14]. The analogy between those transitions goes, in fact, rather far. Dynamically, all of them are induced by the topologically nontrivial classical solutions of underlying theories—vortices in the $O(2)$ model and instantons in QCD₂ respectively, which become statistically dominant at strong coupling [15].

Below we shall first explore the large ϵ limit of the chain model (1.1). This shall be done in the next section. In section 3 we shall construct a family of exact solutions describing the matrix chain in the interval $\epsilon > \epsilon_{\text{cr}}$. The character expansion methods shall be presented in section 4. Finally, in section 5 we shall confirm our conclusions by studying an explicit $1/\epsilon$ expansion for the case of a pure quartic potential $U(x)$.

2. The Infinite Matrix Chain at Large Lattice Spacings

To begin with, we shall demonstrate how the hydrodynamic representation (1.4)–(1.6) reproduces the solution of the infinite matrix chain in the limit $\epsilon \gg \epsilon_{\text{cr}}$. We shall use this special case to understand certain properties of the model—especially the analytic structure of the midway eigenvalue density $\rho_{1/2}(x)$ —which turn out to hold more generally for any finite $\epsilon > \epsilon_{\text{cr}}$.

For large ϵ the “kinetic” term in the matrix model action, $\text{tr}(M_{n+1} - M_n)^2/2\epsilon$ is much smaller than the potential energy term $\epsilon \text{tr}V(M_n)$. Equivalently, one can neglect the nearest neighbor coupling term $\text{tr}(\mathcal{M}_n \mathcal{M}_{n+1})$ in (1.2) thereby reducing \mathcal{Z} to a product of identical one-matrix partition functions

$$\mathcal{Z} = \prod_{n=-\infty}^{+\infty} \int d\mathcal{M}_n \exp \left[-N \text{tr}U(\mathcal{M}_n) \right]. \quad (2.1)$$

The eigenvalue density $\rho(\lambda)$ for the factorized ensemble (2.1) is the same as the density in a one-matrix model with the potential $U(\mathcal{M})$ and can be found from the Riemann–Hilbert equation [16]

$$\oint \frac{\rho(\lambda') d\lambda'}{\lambda - \lambda'} = \frac{1}{2}U'(\lambda). \quad (2.2)$$

For concreteness let us consider the specific case of a quartic double-well potential

$$V(M) = \frac{m^2}{2}M^2 + \frac{\tilde{g}}{4}M^4 \quad (2.3)$$

with $m^2 < 0$ and $\tilde{g} > 0$. By rescaling M, \tilde{g} and the lattice spacing ϵ it is always possible to make the absolute value of m^2 anything we want. From now on we shall choose $m^2 = -4$. As we shall see later, for this m^2 the position of the Kosterlitz–Thouless point is fixed at $\epsilon_{\text{cr}} = 1$. The potential U corresponding to such V has the form

$$U(x) = -\frac{\mu^2 x^2}{2} + \frac{g x^4}{4} \quad (2.4)$$

with $\mu^2 = 2(2\epsilon^2 - 1)$ and $g = \tilde{g}\epsilon^3$.

Strictly speaking, the one-matrix model with a double-well $V(M)$ describes not the $c = 0$ but $c = -2$ quantum gravity. To represent the pure two-dimensional quantum gravity one must change the overall sign of V taking $m^2 > 0$ and $\tilde{g} < 0$. The computational methods presented below are equally applicable in both cases but the formulas happen to be simple for the double-well model. Therefore we shall discuss the double-well $c = -2$ theory first and then outline the modifications arising when $c = 0$.

For the potential $U(M)$ given by (2.4) the solution of the Riemann–Hilbert equation (2.2) is quite easy to find directly. At sufficiently large g (that is, in the strong coupling phase) such solution is given by

$$\pi\rho(x) = (Ax^2 + B)\sqrt{1 - C^2x^2} \quad (2.5)$$

where the coefficients A , B and C equal

$$\begin{cases} C^2 = \frac{1}{8}(\sqrt{\mu^4 + 12g} - \mu^2) \\ B = \frac{1}{6C}(\sqrt{\mu^4 + 12g} - 2\mu^2) \\ A = \frac{g}{2C}. \end{cases} \quad (2.6)$$

Equation (2.5) predicts the eigenvalue density at $x = 0$ to be $\rho(0) = B/\pi$, thereby implying $B > 0$. This condition is satisfied only for $g > g_{\text{cr}} = \mu^4/4$. When the coupling g decreases below $\mu^4/4$ the one-matrix model undergoes a phase transition and the eigenvalue density $\rho(x)$ develops a two-cut structure,

$$\pi\rho(x) = \frac{gx}{2}\sqrt{(x^2 - b^2)(a^2 - x^2)} \quad (2.7)$$

with

$$\begin{cases} a^2 = \frac{1}{g}(\mu^2 + 2\sqrt{g}) \\ b^2 = \frac{1}{g}(\mu^2 - 2\sqrt{g}). \end{cases} \quad (2.8)$$

At the critical point, where $g = g_{\text{cr}} = \mu^4/4$ the solutions for the strong and weak couplings coincide. They yield the eigenvalue density which vanishes at $x = 0$,

$$\pi\rho_{\text{cr}}(x) = \frac{\mu^3x^2}{2\sqrt{2}}\sqrt{1 - \frac{\mu^2x^2}{8}}. \quad (2.9)$$

For small x this critical density behaves as $\rho_{\text{cr}}(x) \propto |x|^\delta$ with $\delta = 2$. The value of the critical index δ is a universal property of the one-matrix model—it does not change even when we modify the matrix model potential by any generic higher order terms. In fact, δ is related to the so-called string susceptibility $\gamma_{\text{str}} = 1 - \delta$, a quantity which plays an important role in string theory.

Let us see how the critical density, and hence the universal property $\rho_{\text{cr}}(x) \propto x^2$, can be extracted from the hydrodynamic picture. The Euler equations which govern the motion of the one-dimensional fluid are

$$\begin{cases} \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}[\rho v] = 0 \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \end{cases} \quad (2.10)$$

where P is the fluid pressure, ρ —its density and v —the velocity. For the special case when P and ρ are related by (1.4) these equations can be simplified and even integrated. To this end, introduce the complex valued function $f = v + i\pi\rho$. In terms of f the system (2.10) reduces to the complex Hopf equation

$$\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} = 0 \quad (2.11)$$

which has the following formal solution

$$f(x, t) = f_0[x - tf(x, t)]. \quad (2.12)$$

The function $f_0(x) = f(x, t = 0)$ represents the requisite initial data for the Hopf equation. In our problem, one specifies only the real part of f_0 given by (1.5) while the imaginary part $\text{Im } f_0(x, 0) = \pi\rho(x)$ must be calculated to satisfy the boundary conditions (1.6).

Due to the time reversal symmetry of Euler equations, imposing (1.6) is equivalent to demanding that the velocity at $t = 1/2$ vanishes identically, $v(x, t = 1/2) \equiv 0$. Therefore, the value of the Hopf function $f(x, t)$ at $t = 1/2$ ought to be purely imaginary, $f(x, 1/2) = i\pi\rho_{1/2}(x)$. As a result, the complex equation

$$i\pi\rho_{1/2}(x) = f_0[x - \tfrac{1}{2}i\pi\rho_{1/2}(x)] \quad (2.13)$$

must have a real-valued solution for $\rho_{1/2}(x)$.

In the limit of large ϵ the critical density (2.9) and, more generally, (2.5) or (2.7), do give rise to a real $\rho_{1/2}$. For example, if $\rho(x) = \rho_{\text{cr}}(x)$ the corresponding initial Hopf function equals

$$\begin{aligned} f_0(z) &= v(z) + i\pi\rho_{\text{cr}}(z) \\ &= -z - \frac{\mu^2}{2}z + \frac{\mu^4}{8}z^3 - \frac{\mu^4 z^3}{8} \sqrt{1 - \frac{8}{\mu^2 z^2}} \end{aligned} \quad (2.14)$$

where we have used (1.5), substituted $g = g_{\text{cr}} = \mu^4/4$ and continued $f_0(z)$ to $z > 2\sqrt{2}/\mu$. Now, we shall see in a moment that the typical values of $z = x - i\pi\rho_{1/2}(x)/2$ appearing in equation (2.13) are of order one, $z \sim \mathcal{O}(1) \gg 1/\mu$. Therefore, we can expand $f_0(z)$ for large μ and finite z ,

$$f_0(z) = -z + \frac{1}{z} + \mathcal{O}(\mu^{-2}) \quad (2.15)$$

which, together with (2.13) implies

$$\pi\rho_{1/2}(x) = 2\sqrt{1 - x^2} + \mathcal{O}(\mu^{-2}). \quad (2.16)$$

A careful inspection of the above computation shows that equation (2.16) holds for any eigenvalue density satisfying the Riemann–Hilbert equation (2.2). Actually, it is simply a consequence of the normalization condition $\int \rho(x) dx = 1$. Indeed, it is easy to compute $f_0(z)$ given any general $U(x)$,

$$\begin{aligned} f_0(z) &= \frac{1}{2}U'(z) - z + i\pi\rho(z) \\ &= -z + \int \frac{\rho(y) dy}{z - y}. \end{aligned} \tag{2.17}$$

For an eigenvalue distribution localized around zero in a small interval of width $\propto 1/\mu$ this formula can be expanded in $1/z$ to yield (2.15) and, consequently, (2.16).

As a result, the shape of $\pi\rho_{1/2}(x)$ in the large μ approximation says very little about the properties of the underlying matrix model. In fact, if we started with $\pi\rho_{1/2}(x) = 2\sqrt{1-x^2}$ at $t = 1/2$ and let it evolve until $t = 1$ according to the Euler equations, that distribution would have collapsed to a delta function. The actual properties of the matrix chain are therefore encoded not in the semicircular shape of (2.16) but rather in tiny perturbations over it. These perturbations are strongly amplified as the distribution collapses and produce the nontrivial eigenvalue density $\rho(x, t = 1) = \rho(x)$ at the final point of time evolution.

Quite remarkably, the effect of small perturbations becomes very transparent if we consider the analytic structure of $\rho_{1/2}(x)$. One discovers that $\rho_{1/2}$ has, in general, several different regular branches. Equation (2.16) approximates, of course, only one of them—the “physical” branch. However, for $\epsilon > \epsilon_{\text{cr}}$ it happens to be another “hidden” branch that actually controls the shape of $\rho(x) \equiv \rho(x, t = 1)$. To see this take $f_{1/2}(x) = i\pi\rho_{1/2}(x)$ as the initial condition for the Hopf equation and evolve it backwards in time thus obtaining $f(x, t = 0) = f_0(x)$. The functions $f_{1/2}$ and f_0 are related by a formula analogous to (2.13)

$$f_0(x) = f_{1/2}\left[x + \frac{1}{2}f_0(x)\right]. \tag{2.18}$$

Now we can substitute $x = -iy$ and use the explicit form of $f_0(x)$ from (2.14) to derive the following parametric representation for $\pi\rho_{1/2}$

$$\begin{cases} \pi\rho_{1/2}(i\xi) = F(y) \\ \xi = -y + \frac{1}{2}F(y) \end{cases} \tag{2.19}$$

where

$$F(y) = y + \frac{\mu^2}{2}y + \frac{\mu^4}{8}y^3 - \frac{\mu^4}{8}y^2\sqrt{y^2 + \frac{8}{\mu^2}}. \tag{2.20}$$

Let us emphasize that this representation applies to the special (but most interesting) case when the matrix model coupling g has been tuned to the critical value $g = g_{\text{cr}}(\epsilon)$.

The plot of $\pi\rho_{1/2}(i\xi)$ as a function of ξ is shown in fig. 2 and clearly reveals two distinct branches. For small ξ one of them behaves as $\pi\rho_{1/2}(i\xi) \approx 2\sqrt{1+\xi^2}$ which is simply an analytic continuation of the physical branch (2.16). The values of the parameter y corresponding to this branch are of order one, $y \sim 1 \gg 1/\mu$, so that $F(y) = y + 1/y + \mathcal{O}(\mu^{-2})$.

Another branch of $\pi\rho_{1/2}$ —call it the “hidden” branch—is obtained when $y \ll 1/\mu$. Then for large μ we have $F(y) \approx y(1 + \mu^2/2)$ and therefore

$$\pi\rho_{1/2}(i\xi) = 2\xi\left(1 + \frac{4}{\mu^2}\right) + \dots \quad (2.21)$$

To demonstrate the importance of this hidden branch let us imagine that we have expanded the critical eigenvalue density (2.9) in powers of x . As it turns out, such an expansion is in one-to-one correspondence with the expansion of the hidden branch in powers of ξ . That is to say, given a power series for $\pi\rho_{1/2}(i\xi)$ in (2.21) and using equation (2.18) one can restore, order by order, the power series for $\rho_{\text{cr}}(x)$.

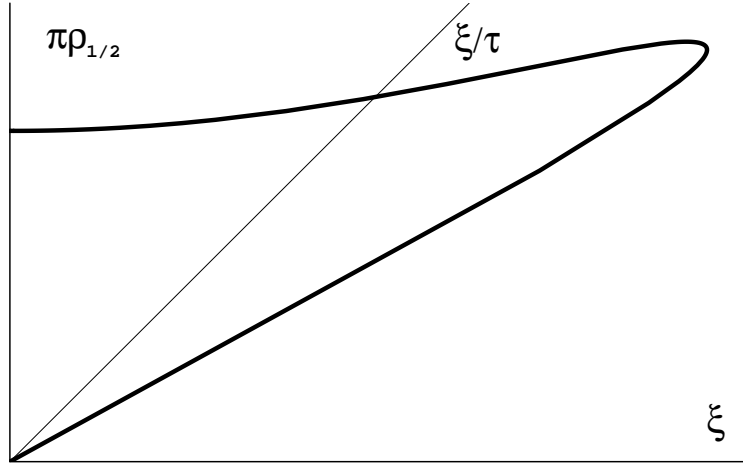


Fig. 2: The midway density $\pi\rho_{1/2}(i\xi)$ for imaginary arguments $x = i\xi$, and the graphic solution of equation $\pi\rho_{1/2}(i\xi) = \xi/\tau$. The plot clearly shows two branches of $\pi\rho_{1/2}$.

The same would not by any means be true for the semicircular physical branch (2.16). There an expansion around zero has nothing to do with how $\rho_{\text{cr}}(x)$ behaves at small x . From this point of view, it is quite satisfactory that the complicated and nonlocal perturbations of the physical branch can be summarized in a simple Taylor expansion of another branch.

Of course, to get the particular graph shown in fig. 2 we have used the one-matrix model limit $\epsilon \gg 1$. However, we shall see in the next section that the plot of $\pi\rho_{1/2}(i\xi)$ remains qualitatively the same for any $\epsilon > 1$. Therefore, the conclusions derived from such a plot—say, the existence of two branches for $\pi\rho_{1/2}(i\xi)$ —are not at all restricted to the large ϵ limit.

For a finite ϵ the hidden branch of $\pi\rho_{1/2}(i\xi)$ is no longer approximated by (2.21). Instead, it has a certain—yet unknown—Taylor expansion

$$\pi\rho_{1/2}(i\xi) = a_1(\epsilon)\xi + a_2(\epsilon)\xi^2 + a_3(\epsilon)\xi^3 + \dots \quad (2.22)$$

all the Taylor coefficients being functions of ϵ . As we advertized above, this expansion can be used, together with (2.18), to reconstruct the potential $U(x)$ and the eigenvalue density $\rho(x) = \rho(x, t = 0)$. To this effect we consider the expansion

$$f_0(x) = f_1x + f_2x^2 + f_3x^3 + \dots \quad (2.23)$$

and solve (2.18) order by order in powers of x . This yields

$$\begin{cases} f_1 = -\frac{2a_1(\epsilon)}{a_1(\epsilon) - 2} \\ f_2 = i\frac{8a_2(\epsilon)}{[a_1(\epsilon) - 2]^3} \\ f_3 = \frac{32a_2^2(\epsilon)}{[a_1(\epsilon) - 2]^5} - \frac{16a_3(\epsilon)}{[a_1(\epsilon) - 2]^4} \end{cases} \quad (2.24)$$

and so forth. The real part of $f_0(x)$ is related, by virtue of (1.5) and (2.4), to the parameters μ^2 and g . In particular, $f_1 = -(\mu^2/2 + 1) = -2\epsilon^2$ which fixes $a_1(\epsilon)$ in terms of ϵ

$$a_1(\epsilon) = \frac{2\epsilon^2}{\epsilon^2 - 1}. \quad (2.25)$$

We see that the hidden branch develops a singularity and ceases to exist at $\epsilon = \epsilon_{\text{cr}} = 1$. This is precisely the location of the Kosterlitz–Thouless phase transition in the infinite matrix chain with the potential (2.4). Remarkably, the same critical value of ϵ_{cr} can be derived independently by studying the singularities of the $\epsilon < 1$ phase and by a variety of other methods [5][6][12].

Therefore, the two-branch structure is a characteristic property distinguishing the $\epsilon > 1$ phase of the infinite matrix chain. It contrasts sharply with the one-branch behavior found in the $\epsilon < 1$ phase of the same theory [12]. Indeed, for $\epsilon < 1$ the small x properties

of the critical eigenvalue density can be restored from the physical $\pi\rho_{1/2}(x)$ alone, without continuing it to any other branches.

The presence of the second branch has a very interesting reflection in how the droplet evolves with time. It results in the formation of a shock—a state where the spatial derivative of the fluid density becomes infinite at a certain point. To see how this happens consider the time dependence of the droplet density at the origin $\rho(x=0, t) \equiv d(t)/\pi$. We can determine the function $d(t)$ from an implicit relation analogous to (2.12)

$$f(x, t) = f_{1/2}[x + \tau f(x, t)] \quad (2.26)$$

τ being the shifted time $\tau = 1/2 - t$. Setting here $x = 0$ and remembering that $v(x = 0, t) = 0$ at any t , we obtain a transcendental equation on $d(t)$

$$d(t) = \pi\rho_{1/2}[i\tau d(t)]. \quad (2.27)$$

This equation can be easily solved graphically using the plot of $\pi\rho_{1/2}(i\xi)$ shown in fig. 2. One simply writes $\xi = \tau d(t)$ so that

$$\pi\rho_{1/2}(i\xi) = \frac{\xi}{\tau} \quad (2.28)$$

and finds the intersection point of the plot for $\pi\rho_{1/2}(i\xi)$ with the straight line of the slope $1/\tau$. At $t = 1/2$, when $\tau = 0$, this yields a nonzero d given by the value of the physical branch at $x = 0$. As we go backwards in time away from $t = 1/2$, the fluid density at the origin changes. In the specific case depicted in fig. 2 it first grows and then starts decreasing until, at a certain moment $\tau = \tau_{\text{cr}}$, the density at $x = 0$ vanishes. This critical time τ_{cr} is determined by the slope of the hidden branch at the origin

$$\frac{1}{\tau_{\text{cr}}} = \left. \frac{d\pi\rho_{1/2}(i\xi)}{d\xi} \right|_{\xi=0}. \quad (2.29)$$

For any $\tau > \tau_{\text{cr}}$ or, equivalently, for $t < t_{\text{cr}} = 1/2 - \tau_{\text{cr}}$ equation (2.28) admits only the solution $d(t) = 0$. Note that, as a consequence of (2.25), the critical time is a function of the lattice spacing $t_{\text{cr}} = 1/(2\epsilon^2)$.

As a result, the droplet density at the origin remains strictly zero at $0 \leq t \leq t_{\text{cr}}$ and becomes positive in the interval $t_{\text{cr}} < t \leq 1/2$. Of course, the densities at larger times $t \in [1/2, 1]$ are easily obtained using the time reflection symmetry $d(t) = d(1 - t)$. The shock-type configuration forms at $t = t_{\text{cr}}$ and $t = 1 - t_{\text{cr}}$. This can be understood by solving equation (2.26) for small, but nonzero, values of x and times t close to t_{cr} . At such t the graphical solution to equation (2.28) yields a value of ξ very close to zero. Consequently,

we can use the Taylor expansion of $\pi\rho_{1/2}(i\xi)$, given by (2.22), as input for equation (2.26) thereby obtaining

$$\rho(x, t) \propto \begin{cases} a_2(\epsilon) \left(\frac{\tau_{\text{cr}}}{\tau - \tau_{\text{cr}}} \right)^3 x^2 & \text{when } t < t_{\text{cr}} \\ \sqrt{\frac{a_1^3(\epsilon)}{2a_2(\epsilon)}} x & \text{when } t = t_{\text{cr}} \\ \frac{\tau_{\text{cr}} - \tau}{a_2(\epsilon)\tau^2\tau_{\text{cr}}} + a_2(\epsilon) \left(\frac{\tau_{\text{cr}}}{\tau_{\text{cr}} - \tau} \right)^3 x^2 & \text{when } t > t_{\text{cr}} \end{cases} \quad (2.30)$$

for $x \rightarrow 0$ and $|t - t_{\text{cr}}| \ll 1$. A typical plot of $\rho(x, t)$ at various times is shown in fig. 3. When $t < t_{\text{cr}}$ the fluid flows smoothly towards $x = 0$, its velocity increasing in time. While t approaches t_{cr} , more and more fluid keeps coming in, causing the density gradient $(\partial/\partial x)\rho(x = 0, t_{\text{cr}})$ to blow up. This shows as a cusp in the plot of $\rho(x, t_{\text{cr}})$. After that, the fluid starts accumulating at $x = 0$, and the velocity $v(x, t)$ becomes smooth again.

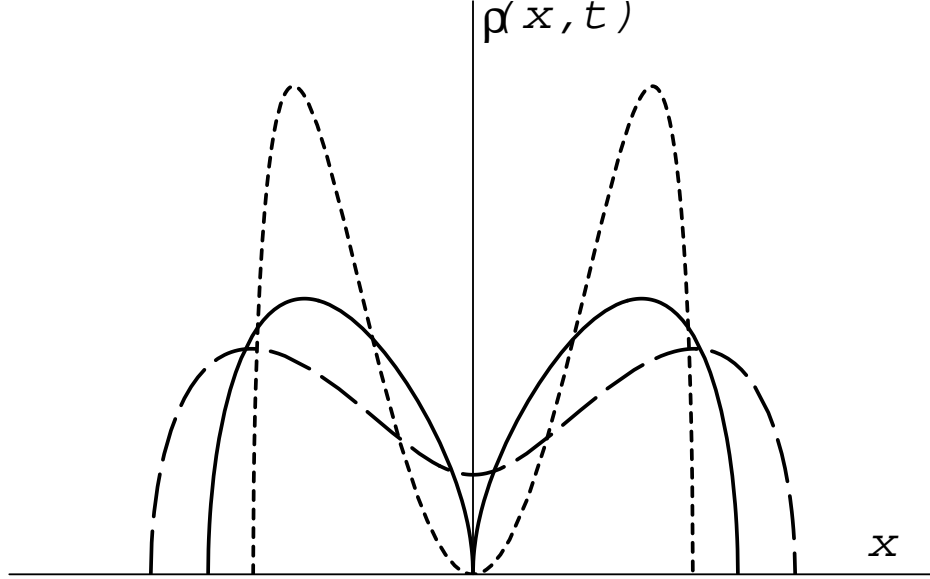


Fig. 3: The droplet density $\rho(x, t)$ at various stages of evolution: $t < t_{\text{cr}}$ (dotted line), $t = t_{\text{cr}}$ (solid line) and $t > t_{\text{cr}}$ (dashed line). The plot of $\rho(x, t_{\text{cr}})$ exhibits a “shock” at $x = 0$.

Nothing similar ever occurs in the $\epsilon < 1$ phase. There at $g = g_{\text{cr}}(\epsilon)$ the droplet density behaves as $\rho(x, t) \propto \alpha(t)|x| + \mathcal{O}(x^2)$ with $\alpha(t)$ being an infinitely smooth function of time. In that phase $\rho(x = 0, t) \equiv 0$ for all $t \in [0, 1]$.

One could say that the quantity $\zeta = \rho_{1/2}(0)$ is an “order parameter” for our theory. Indeed, $\zeta \equiv 0$ identically at any $\epsilon < 1$. On the other hand, when $\epsilon > 1$ the values of ζ are

manifestly positive. Although such an order parameter does not describe the spontaneous breaking of any symmetry, it does parametrize a change in the group representation structure of our model. This shall be demonstrated in section 4. But before that let us use the intuition derived from the large ϵ limit to construct an exact solution of the infinite matrix chain at $\epsilon > 1$.

3. Exact Solutions of the Matrix Chain for Lattice Spacings Greater than One

In this section we shall obtain a class of potentials $U(x)$ for which the chain model can be solved exactly. We shall see that the critical index δ corresponding to these potentials equals $\delta(\epsilon) = 2$ at any ϵ between one and infinity.

The solutions presented below are not as explicit as exact solutions are usually expected to be. Neither the interaction potentials nor the eigenvalue densities corresponding to these solutions can be written in terms of elementary or special functions. Instead, they are determined by complicated transcendental equations, although the critical indices for the “transcendental” solutions are still calculable explicitly.

In principle, to produce such exact solutions is not hard. Choose an arbitrary, but properly normalized $\rho_{1/2}(x)$. Imagine a droplet which at $t = 1/2$ is at rest and has this density. Evolve this droplet backwards in time up to $t = 0$ and read off the fluid velocity $v(x) = v(x, t = 0)$ together with the density $\rho(x) = \rho(x, t = 0)$. The result is an exact solution—given by $\rho(x)$ —to the matrix chain model where the potential $U(x)$ can be found from (1.5).

The most difficult part of this program is to adjust $\rho_{1/2}(x)$ so that the potential $U(x)$ comes out physically reasonable. Let us require that $U(x)$ be a smooth and bounded from below double-well potential with a hump at $x = 0$,

$$U(x) = -\frac{\mu^2}{2}x^2 + \mathcal{O}(x^4). \quad (3.1)$$

Here, as before, $\mu^2 = 2(2\epsilon^2 - 1)$.

The potentials in this class are qualitatively similar to the pure quartic potential (2.4). Therefore, due to the phenomenon of universality, we can expect that the critical exponents of the chain model with these potentials are strictly equal to the critical exponents of the pure quartic model. This hypothesis shall be confirmed in section 5 where we investigate the pure quartic case using a systematic strong coupling expansion.

A carelessly chosen $\rho_{1/2}$ would most likely yield the potential with $\mu^2 < 2$, which corresponds to $\epsilon < 1$. To generate $U(x)$ with $\mu^2 > 2$ the analytic structure of $\rho_{1/2}$ must be

very special. This structure can be inferred from the results for the $\epsilon \gg 1$ limit obtained in section 2. That is, let us choose $\rho_{1/2}(x)$ so that

- (a) as an analytic function of x , it has at least two branches (not counting the branches different only by a minus sign);
- (b) one of these branches is an even function of x , real valued for real x (this shall be the “physical” branch representing the actual droplet density at $t = 1/2$);
- (c) the “physical” branch is normalized, $\int \rho_{1/2}(x) dx = 1$;
- (d) the second branch of $\rho_{1/2}$ has the Taylor expansion given by (2.22).

Although these requirements are quite restrictive, there are many functions obeying all of the conditions (a)–(d). As an example, consider the function $r(x) = [\pi\rho_{1/2}(x)]^2$ specified by the following parametric representation

$$\begin{cases} r(i\xi) = R^2(\varphi) \cosh(\kappa\varphi) \\ \xi = R(\varphi) \sinh(\kappa\varphi) \end{cases} \quad (3.2)$$

where $\varphi \in [0, \pi/2]$ and $R(\varphi)$ is an even analytic function of φ such that $R(\pi/2) = 0$. For concreteness, let us choose

$$R(\varphi) = \sum_{n=1}^{n_{\max}} R_n \cos^n \varphi \quad (3.3)$$

with arbitrary real coefficients R_n .

It is easy to check that the plot of $r(i\xi)$ has two branches as in fig. 3 and that the “physical” branch of r is even. Furthermore, the Taylor expansion of the “hidden” branch has the form which agrees precisely with (2.22),

$$r(i\xi) = \xi^2 \frac{\cosh(\pi\kappa/2)}{\sinh^2(\pi\kappa/2)} \left\{ 1 + \xi \frac{\kappa}{R_1} \frac{\cosh(\pi\kappa/2)}{\sinh^2(\pi\kappa/2)} \left[2 - \tanh^2\left(\frac{\pi\kappa}{2}\right) \right] + \mathcal{O}(\xi^2) \right\} \quad (3.4)$$

so that condition (d) is also satisfied. The values of the physical branch at real x can be found from (3.2) using the substitution $\varphi = -i\eta$,

$$\begin{cases} r(x) = \cos(\kappa\eta) \left[\sum_{n=1}^{n_{\max}} R_n \cosh^n \eta \right]^2 \\ x = \sin(\kappa\eta) \sum_{n=1}^{n_{\max}} R_n \cosh^n \eta. \end{cases} \quad (3.5)$$

As we see, the function $r(x)$ vanishes at $\eta = \pm\pi/2\kappa$. Therefore, the support of $\pi\rho_{1/2}(x) = \sqrt{r(x)}$ is restricted to the finite interval $[-x_{\max}, x_{\max}]$ with $x_{\max} = x(\eta = \pm\pi/2\kappa)$, and therefore the normalization integral $\int \rho_{1/2}(x) dx$ is finite. The value of this integral can

be adjusted to equal one by rescaling all the coefficients $R_n \rightarrow \lambda R_n$ with an appropriately chosen λ .

The potential $U(x)$ which corresponds to (3.2) must be a nonsingular infinitely differentiable function of x . This condition should be enforced at least for those x where $\rho(x) \neq 0$ —otherwise additional spurious singularities in $\rho(x)$ could be produced and the universality argument (the paragraph after (3.1)) would become inapplicable.

Technically, U and ρ are determined by the $t = 0$ values of the Hopf function $f(x, t) = v(x, t) + i\pi\rho(x, t)$ which, in turn, is constrained by equation (2.26). Consequently, f solves a system of transcendental equations obtained from (3.5) by the substitution

$$\begin{cases} x \rightarrow x + \tau f(x, t) \\ r \rightarrow -f^2(x, t). \end{cases} \quad (3.6)$$

Generally, this equation has multiple roots. As $\tau = 1/2 - t$ increases from 0 to 1/2, some of these roots might coalesce producing unwanted singularities in U and then also in ρ . Therefore, the parameters R_n in (3.3) should be chosen so that this does not happen for any $\tau < 1/2$.

The numerical analysis of such transcendental equations indicates (although we do not have a rigorous proof) that we do have some choice of parameters for any $\epsilon > 1$. This immediately allows us to determine the critical index δ in the $\epsilon > 1$ phase of the matrix chain. Indeed, using the Taylor expansion (3.4) we can read off the coefficients

$$\begin{cases} a_1(\epsilon) = \sqrt{\frac{\cosh(\pi\kappa/2)}{\sinh^2(\pi\kappa/2)}} \\ a_2(\epsilon) = \frac{\kappa}{2R_1} \left[\frac{\cosh(\pi\kappa/2)}{\sinh^2(\pi\kappa/2)} \right]^{3/2} \left[2 - \tanh^2\left(\frac{\pi\kappa}{2}\right) \right]. \end{cases} \quad (3.7)$$

From (2.25) we now deduce that the parameter κ is a function of ϵ ,

$$\frac{\cosh(\pi\kappa/2)}{\sinh^2(\pi\kappa/2)} = \left(\frac{2\epsilon^2}{\epsilon^2 - 1} \right)^2 \quad (3.8)$$

and that for any lattice spacing $a_2(\epsilon) > 0$. Consequently, the eigenvalue density in the chain model behaves as $\pi\rho(x) \equiv \text{Im } f(x, t = 0) \propto f_2 x^2$, where $f_2 > 0$, yielding the eigenvalue index in the $\epsilon > 1$ phase $\delta(\epsilon) = 2$.

By replacing $\cosh(\kappa\varphi)$ and $\sinh(\kappa\varphi)$ in (3.2) with more complicated functions it is possible to obtain models where at certain discrete values of the lattice spacing ϵ the coefficient $a_2(\epsilon)$ vanishes. At these ϵ such models exhibit higher order multicritical behavior.

Of course, the existence of multicritical points in the infinite matrix chain should not come as a surprise. Indeed, exactly the same spectrum of critical indices occurs in one-matrix models which describe the $\epsilon \gg 1$ limit of the chain theory.

In the case of the upside-down quartic potential $V(M) = m^2 M^2/2 + \tilde{g} M^4/4$ with $m^2 > 0$ and $\tilde{g} < 0$ the $\epsilon \gg 1$ limit of the matrix chain becomes a $c = 0$ theory characterized by $\delta = 3/2$. Using the arguments of this and the previous sections it is possible to show that there exist regular potentials leading to $\delta(\epsilon) = 3/2$ at any $\epsilon > 1$. Again, the midway density $\rho_{1/2}(x)$ develops a second branch when the lattice spacing ϵ gets bigger than one, while only one branch enters the picture for $\epsilon < 1$. However, to see the second branch one should continue $\rho_{1/2}(x)$ not to imaginary $x = i\xi$ but rather to real positive very large $x \gg 1/m$. Since for real x the density $\rho_{1/2}(x)$ must always be real, the corresponding “hidden” branch can only have half-integer singularities $\rho_{1/2}(x) \propto (x - x_0)^\delta$ with $\delta = n + 1/2$, the lowest nontrivial value of δ being $\delta = 3/2$.

4. The Character Expansion and the Order Parameter

In this section we shall develop the character expansion for the chain model. We shall find that the $SU(N)$ representation dominating the chain partition function at large N develops a “gap” at the Kosterlitz–Thouless phase transition point, and that the gap size is a simple function of the order parameter $\zeta = \rho_{1/2}(0)$.

There are both conceptual and technical reasons why the character expansion is useful in the matrix chain problem. Conceptually, it shows that the Kosterlitz–Thouless phase transition and the Douglas–Kazakov-type transitions in two-dimensional QCD and the dually weighted graph models all occur through essentially the same mechanism. On a technical level, there is some evidence, based on the singularity structure of the matrix chain with $\epsilon < 1$, that the exact solution of the chain model with a pure quartic potential may belong to the class of elliptic functions [12]. If this is true, the character expansion could be just the right computational tool to derive such a solution.

To construct the character expansion for the infinite matrix chain, let us go back to the partition function (1.2) and diagonalize each matrix $\mathcal{M}_n = U_n \Lambda_n U_n^\dagger$ with diagonal $\Lambda_n = \text{diag}(\lambda_{1,n} \dots \lambda_{N,n})$. Then, express the Hermitian matrix measure $d\mathcal{M}_n$ in terms of U ’s and lambdas,

$$d\mathcal{M}_n = \Delta^2(\lambda_n) d\lambda_{1,n} \dots d\lambda_{N,n} dU_n$$

where $\Delta(\lambda_n)$ is the Van der Monde determinant of eigenvalues $\lambda_{i,n}$ and dU_n refers to the Haar measure on the unitary group $SU(N)$. Furthermore, it is useful to introduce the matrices $V_n = U_{n+1}^\dagger U_n$.

Given this notation the partition function of the infinite matrix chain can be written in the form

$$\mathcal{Z} = \int \prod_{n \in \mathbb{Z}} \Delta^2(\lambda_n) d\lambda_{1,n} \dots d\lambda_{N,n} dU_n \exp \left\{ N \operatorname{tr} \sum_{n=-\infty}^{\infty} \left[V_n \Lambda_n V_n^\dagger \Lambda_{n+1} - U(\Lambda_n) \right] \right\}. \quad (4.1)$$

The matrices V_n represent the angular degrees of freedom so typical of all multimatrix models. Fortunately, since the one-dimensional lattice does not have closed loops, all V_n are mutually independent and we can easily integrate them out. To do this one simply changes variables from $\{U_n\}$ to $\{V_n\}$ with the result

$$\mathcal{Z} = \int \prod_{n \in \mathbb{Z}} \Delta^2(\lambda_n) d\lambda_{1,n} \dots d\lambda_{N,n} dV_n \exp \left\{ N \operatorname{tr} \sum_{n=-\infty}^{\infty} \left[V_n \Lambda_n V_n^\dagger \Lambda_{n+1} - U(\Lambda_n) \right] \right\}. \quad (4.2)$$

Note that no Jacobian arises when we pass from U to V .

In the large N limit the partition function (4.2) can be simplified even further. Indeed, imagine that all integrals over V_n have been done. Then for $N \rightarrow \infty$ the remaining integrals over the eigenvalues are dominated by an n -independent saddle point $\lambda_{i,n} \equiv \lambda_i$. Remarkably, the same saddle point values of λ_i occur in a simpler integral

$$\mathcal{Z}_{\text{EK}} = \int dV \Delta^2(\lambda) d\lambda_1 \dots d\lambda_N \exp \left\{ N \operatorname{tr} [V \Lambda V^\dagger \Lambda - U(\Lambda)] \right\} \quad (4.3)$$

where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$. To prove this assume

$$\int dV \exp [N \operatorname{tr} (V \Lambda V^\dagger \Lambda')] = \exp [N^2 F(\Lambda, \Lambda')]$$

with a certain $F(\Lambda, \Lambda')$, and write down the saddle point equations for both integrals. These equations happen to be the same, and so are their solutions. In fact, the equivalence between \mathcal{Z} and \mathcal{Z}_{EK} is just a very special case of the Eguchi–Kawai reduction theorem [17].

There are several ways to integrate out the matrix V in (4.3). One of them—based on the large N asymptotics of the famous Itzykson–Zuber integral [11][18]—leads eventually to the hydrodynamic picture (1.4)–(1.6). Another useful method is the character expansion [14]. Instead of trying to do the integral over V explicitly, we expand the integrand in a series of $SU(N)$ characters and then use the formula

$$\int dV \chi_R(V A V^\dagger B) = \frac{1}{d_R} \chi_R(A) \chi_R(B). \quad (4.4)$$

Here $\chi_R(U)$ stands for the character of a unitary matrix U in the representation R ; d_R is the dimension of that representation, while A and B are arbitrary matrices. The final output of a character expansion is a sum over representations of $SU(N)$ which can usually be done by the saddle point method [14].

When dealing with large N limits, it is often more convenient to expand in the characters of $U(N)$ rather than $SU(N)$. The $U(N)$ representations are labelled by N integers $n_1 \geq \dots \geq n_N$, the character of a matrix \mathcal{M} in any such representation being given by the Weyl formula

$$\chi_R(\mathcal{M}) = \frac{\det \|\lambda_p^{l_q}\|}{\Delta(\lambda)}. \quad (4.5)$$

As usual, the numbers $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of the matrix \mathcal{M} , whereas the (strictly ordered) integers l_i equal $l_i = n_i + N - i$.

To compute the integral (4.3) let us expand the term containing $W = V\Lambda V^\dagger \Lambda$ in a character series [18]

$$e^{N \operatorname{tr} W} = \sum_R c_R \chi_R(W). \quad (4.6)$$

The coefficients of this expansion are easily found from the orthogonality of characters,

$$\begin{aligned} c_R &= \int dV e^{N \operatorname{tr} V} \chi_R^*(V) \\ &= \frac{\Delta(l)}{l_1! \dots l_N!} N! N^{(l_1 + \dots + l_N) - N(N-1)/2}. \end{aligned} \quad (4.7)$$

Now we can apply (4.4) to do the unitary group integration in (4.3). Throwing out a constant overall factor and using the formula for the dimension of a representation $R = \{l_1, \dots, l_N\}$

$$d_R = \prod_{i < j} \frac{l_i - l_j}{j - i} \quad (4.8)$$

this finally yields

$$\mathcal{Z}_{\text{EK}} = \sum_{l_1, \dots, l_N=0}^{\infty} \frac{N^{l_1 + \dots + l_N}}{l_1! \dots l_N!} \int d\lambda_1 \dots d\lambda_N \Delta^2(\lambda) \chi_R^2(\Lambda) e^{-N \operatorname{tr} U(\Lambda)}. \quad (4.9)$$

At large N both the sum over l_1, \dots, l_N and the integral over the eigenvalues in (4.9) are dominated by saddle points. Formally, these saddle points are defined as the values of l_1, \dots, l_N and $\lambda_1, \dots, \lambda_N$ that maximize the integrand of (4.9). In fact, the saddle point for $\{\lambda_i\}$ is already known to us. It is given by the same eigenvalues that dominate the original matrix chain integrand (1.2) at $N \rightarrow \infty$ and which are described by the density $\rho(\lambda)$ in (1.6).

At the saddle point in the representation space the values of l_i are all of order N . Therefore, it shall be convenient to introduce the quantities $h_i = l_i/N$ which, in the large N limit, condense to form a smooth distribution $\rho_l(h)$. Since all of the l 's are different from each other, this density can never be greater than one.

To maximize the integrand of (4.9) with respect to l_1, \dots, l_N we first use the Stirling formula $N^{l_i}/l_i! \approx \exp[Nh_i(1 - \log h_i)]$. Also, the large N asymptotics of the character $\chi_R(\Lambda)$ is given by $\chi_R(\Lambda) \propto \exp\{N^2 \Xi[\rho_l(h), \rho(\lambda)]\}$ where the functional $\Xi[\rho_l(h), \rho(\lambda)]$ has a finite large N limit. In view of this notation, the saddle point equation for the l 's can be written in the form

$$2 \frac{\partial}{\partial h} \frac{\delta}{\delta \rho_l(h)} \Xi[\rho_l(h), \rho(\lambda)] = \log h. \quad (4.10)$$

The equations needed to compute Ξ are provided by the method of [11]. However, we should be particularly careful as some of the λ_i (in fact, half of them for an even $U(\lambda)$) have a negative sign. Then the sign of an expression like $\lambda_p^{l_q} = \lambda_p^{N h_q}$ oscillates when N increases and its large N behavior is not well defined. To circumvent the difficulty, imagine that $\lambda_1, \dots, \lambda_{N/2}$ are all positive and that the negative ones are just $-\lambda_1, \dots, -\lambda_{N/2}$. Furthermore, assume that among l_i half are odd (call them l_i^o) and half are even (denoted l_i^e .) Certainly, if the set of all (odd and even) l 's converges to form the distribution $\rho_l(h)$, the same distribution would also describe the densities of both $\{l_i^o\}$ and $\{l_i^e\}$ taken separately from each other.

For such symmetric distributions of $\{\lambda_i\}$ and $\{l_i\}$ the Weyl determinant factorizes into a product of two $(N/2) \times (N/2)$ determinants [14]

$$\det \|\lambda_p^{l_q}\| = (-2)^{N/2} \det \|\lambda_{+p}^{l_q^e}\| \det \|\lambda_{+p}^{l_q^o}\| \quad (4.11)$$

where λ_{+p} ($p = 1, \dots, N/2$) are the $N/2$ positive eigenvalues $\lambda_1, \dots, \lambda_{N/2}$. Both of these smaller determinants exhibit identically the same leading large N behavior $\exp\{N_*^2 \mathcal{D}[\rho_l(h), \rho(\lambda)]\}$ with $N_* = N/2$ being the new determinant size.

To find \mathcal{D} we first represent λ_{+p} in the manifestly positive form $\lambda_{+p} = \exp(\theta_p/2)$ so that $\lambda_{+p}^{l_q^e} = \exp(N_* \theta_p h_q^e)$. The density of thetas $\sigma(\theta)$ is, of course, related to $\rho(\lambda_+)$. Since $N_* \sigma(\theta) d\theta = N \rho(\lambda_+) d\lambda_+$, one gets

$$\sigma(\theta) = e^{\theta/2} \rho(e^{\theta/2}). \quad (4.12)$$

Now we can write down the equations which fix \mathcal{D} . Let us introduce the two functions

$$\begin{cases} G_+(\theta) = \frac{\partial}{\partial \theta} \frac{\delta \mathcal{D}}{\delta \sigma(\theta)} + i\pi \sigma(\theta) \\ G_-(h) = \frac{\partial}{\partial h} \frac{\delta \mathcal{D}}{\delta \rho_l(h)} - i\pi \rho_l(h). \end{cases} \quad (4.13)$$

Then the equation on \mathcal{D} is simply [11][19]

$$G_+[G_-(h)] = h. \quad (4.14)$$

This complex-valued functional equation is equivalent to two real-valued constraints which determine the two unknown functions $(\partial/\partial\theta)\delta\mathcal{D}/\delta\sigma(\theta)$ and $(\partial/\partial h)\delta\mathcal{D}/\delta\rho_l(h)$.

On the other hand, the functional derivatives of \mathcal{D} can be extracted from the saddle point equations. In this way, equation (4.10) yields

$$\frac{\partial}{\partial h} \frac{\delta\mathcal{D}}{\delta\rho_l(h)} = \log h. \quad (4.15)$$

We can also replace $\lambda_i \rightarrow \pm \exp(\theta_i/2)$ in (4.9), maximize it with respect to θ_i and get the saddle point equation for the thetas. Remembering that $N = 2N_*$, we derive

$$\frac{\partial}{\partial\theta} \frac{\delta\mathcal{D}}{\delta\sigma(\theta)} = \frac{1}{2} e^{\theta/2} U'(e^{\theta/2}). \quad (4.16)$$

Finally, let us combine (4.13), (4.15) and (4.16) together with equation (4.14). The result can be conveniently expressed in terms of an auxiliary function $H(z) = U'(z)/2 + i\pi\rho(z)$ and reads

$$H\left[\sqrt{h} \exp\left(-\frac{i\pi\rho_l(h)}{2}\right)\right] = \sqrt{h} \exp\left(\frac{i\pi\rho_l(h)}{2}\right). \quad (4.17)$$

Very remarkably, this equation leads to a very simple formula connecting ρ_l to $\rho_{1/2}$. Indeed, from (1.5) and (2.18) we have the following relation between $\rho_{1/2}$ and H ,

$$i\pi\rho_{1/2}\left[\frac{1}{2}(H(x) + x)\right] = H(x) - x. \quad (4.18)$$

Now let us substitute $x = \sqrt{h} \exp[-i\pi\rho_l(h)/2]$ and use (4.17). This produces, at last, the formula

$$\pi\rho_{1/2}\left[\sqrt{h} \cos\left(\frac{\pi\rho_l(h)}{2}\right)\right] = 2\sqrt{h} \sin\left(\frac{\pi\rho_l(h)}{2}\right) \quad (4.19)$$

which is our final result.

Equation (4.19) is essentially kinematical. Indeed, it does not involve the matrix model interaction potential in any way¹. However, it does indicate that the descriptions of the infinite chain model based on the hydrodynamic picture and on the character expansion are completely equivalent.

¹ Equation (4.19) can also be derived in an “explicitly kinematic” fashion following the ideas of Douglas [20].

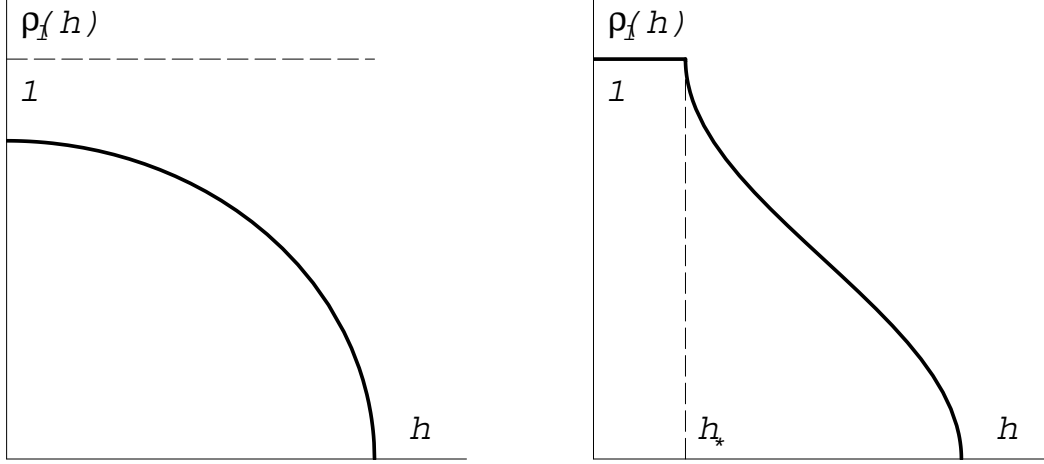


Fig. 4: The Young tableau densities $\rho_l(h)$ for $\epsilon < 1$ (left) and for $\epsilon > 1$ (right). The presence of a gap in the Young tableau for $\epsilon > 1$ shows as a plateau on the graph of $\rho_l(h)$.

To be rigorous, equation (4.19) holds only for those h where $\rho_l(h)$ is strictly less than one. Indeed, if the Young tableau density reaches its upper bound $\rho_l(h) = 1$ the saddle point equation (4.15) ceases to apply and the whole derivation becomes invalid. Practically, this allows us to determine whether there is, in fact, an interval of h where $\rho_l(h) = 1$. If yes, then the size of this interval h_* equals the “gap width” in the corresponding Young diagram as shown in fig. 1.

Since h_* is just on the border of the region where (4.19) applies, we can use that formula with $h = h_*$ and $\rho_l(h_*) = 1$. This yields $\zeta = \pi\rho_{1/2}(0) = 2\sqrt{h_*}$ thereby proving equation (1.7). In other words, the gap size, which appears to be a natural “order parameter” for the phase transition in the character expansion language is related to ζ —an order parameter in the hydrodynamic picture.

Therefore, in the critical regime $h_* \equiv 0$ for $\epsilon < 1$ while $h_* = h_*(\epsilon) > 0$ for any $\epsilon > 1$. This picture of the Kosterlitz–Thouless phase transition is completely identical to the mechanism of the Douglas–Kazakov transition in large N two-dimensional QCD [13][15]. Indeed, both in the matrix chain model and in QCD₂ the density $\rho_l(h)$ develops a plateau for large lattice spacings (respectively, strong coupling) while $\rho_l(h) < 1$ everywhere for small lattice spacings (or weak couplings.) And, to complete the correspondence, both transitions are induced by the topologically nontrivial states—vortices and instantons—that dominate in the strongly coupled phases of these two theories.

5. The Chain Model with a Pure Quartic Potential

In this section we shall construct and explore a systematic strong coupling expansion for the infinite matrix chain with a quartic interaction potential. The expansion we shall consider is in powers of $1/\epsilon^2$ or, equivalently, $1/\mu^2$.

For large μ^2 the successive terms in the character expansion (4.9) become smaller and smaller. Indeed, any character $\chi_R(\Lambda)$ is a homogeneous polynomial in the eigenvalues λ_i . The degree of homogeneity for such a polynomial equals the total number of boxes in the corresponding Young tableau, $n_R = n_1 + \dots + n_N = l_1 + \dots + l_N - N(N-1)/2$. Consequently, if for large μ we approximate the eigenvalue density $\rho(\lambda)$ by the one-matrix model result (2.5), the typical width of such distribution will be $\lambda_{\text{typ}} \sim 1/\mu$ and the value of the character $\chi_R(\Lambda)$ will be suppressed by a factor $1/\mu^{n_R}$. In other words, to expand in $1/\mu$ up to the order $1/\mu^{2n_R}$ we should choose only those representations in (4.9) which have not more than n_R boxes. The characters in such representations shall be polynomials in $\text{tr}\Lambda, \text{tr}\Lambda^2, \dots, \text{tr}\Lambda^{n_R}$ and would effectively generate corrections to the one-matrix potential $U(\lambda)$. That is to say, they will perturb the right hand side of the Riemann–Hilbert equation (2.2) by the terms of order at most λ^{n_R-1} .

A one-matrix model with such modified potential has, in general, the solution²

$$\pi\rho(x) = Q(x)\sqrt{1 - \frac{\mu_{\text{r}}^2 x^2}{8}} \quad (5.1)$$

where $Q(x)$ is a polynomial of degree $n_R - 2$ and μ_{r} —the “renormalized” distribution width [16]. Obviously, if the original potential $U(\lambda)$ was even, so shall be $Q(x)$. The coefficients of Q could, in principle, be determined by computing the characters, evaluating the corrections to $U(\lambda)$ and solving the resulting one-matrix model.

Fortunately, there is a more economical way to organize this computation. Let us use the fact [11][12] (following essentially from (2.12)) that for any infinite chain model the functions

$$\mathcal{G}_{\pm}(z) = \frac{1}{2}U'(z) \pm i\pi\rho(z) \quad (5.2)$$

(with $\rho(z)$ being the exact eigenvalue density) obey the functional equation

$$\mathcal{G}_+[\mathcal{G}_-(z)] = z. \quad (5.3)$$

Very remarkably, this equation allows one to compute $Q(z)$ much easier and quicker.

² Since we are interested in the critical regime, the coupling constant g is always adjusted to equal $g_{\text{cr}}(\epsilon)$. For the uncorrected one-matrix model this would imply $g = \mu^4/4$.

The left hand side of equation (5.3) contains the eigenvalue density at a complex point, $\rho[\mathcal{G}_-(z)]$. Consequently, it is important to choose the correct analytic branch of ρ in both \mathcal{G}_+ and \mathcal{G}_- . Since it is more convenient to work with real variables, we shall enforce equation (5.3) in the region of large positive $z > 2\sqrt{2}/\mu_r$, where $i\pi\rho(z)$ becomes real. The correct sign choice for $\rho(z)$ in that region yields

$$\mathcal{G}_+(z) = \mathcal{G}_-(z) \equiv \mathcal{G}(z) = -\frac{\mu^2}{2}z + \frac{g}{2}z^3 - Q(z)\frac{\mu_r z}{2\sqrt{2}}\sqrt{1 - \frac{8}{\mu_r^2 z^2}}. \quad (5.4)$$

The reason is, $\rho(z)$ has a square root branch cut between $x_{\max} = 2\sqrt{2}/\mu_r$ and $-x_{\max} = -2\sqrt{2}/\mu_r$. If a real z were positioned on the upper edge of this cut the value of $\mathcal{G}_-(z)$ would have a negative imaginary part. In evaluating $\rho[\mathcal{G}_-(z)]$ this would take us below the lower edge. But the values of $\rho(z)$ on the lower edge are opposite to the values of ρ on the upper edge. When z is continued to $z > 2\sqrt{2}/\mu_r$ the branch of $\rho(z)$ entering \mathcal{G}_- should be continued from above the cut while $\rho(z)$ in \mathcal{G}_+ should be continued from below the cut. This compensates for the sign difference in front of $\rho(z)$ in (5.2) and leads to the result (5.4).

At infinitely large μ , which corresponds to the one-matrix model case, we get

$$\mathcal{G}_+(z) = \mathcal{G}_-(z) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{\mu^2}\right) \quad (5.5)$$

so that equation (5.3) is trivially satisfied. The $1/\mu^2$ corrections can now be derived by perturbing this solution. Since we are interested in the critical regime where $\rho(x) \propto |x|^\delta$ the polynomial $Q(z)$ must vanish at $z = 0$, and we can seek a solution of the form

$$\mathcal{G}(z) = -\frac{\mu^2}{2}z + \frac{g}{2}z^3 - z^3 P(z) \frac{\mu_r^4}{8} \sqrt{1 - \frac{8}{\mu_r^2 z^2}}. \quad (5.6)$$

The coupling $g = g_{\text{cr}}(\mu)$ and the coefficients of $P(z)$ should be computed from equation (5.3) together with the normalization requirement $\int \rho(x) dx = 1$.

Technically, it is easier to expand in powers of $1/\mu_r^2$ rather than $1/\mu^2$. This is perfectly legitimate given that, to the leading order in our large μ expansion, $\mu = \mu_r$. In other words, we shall seek corrections to the one-matrix model relations $\mu = \mu_r$ and $g_{\text{cr}} = \mu_r^4/4$,

$$\begin{cases} \mu^2 = \mu_r^2 \left(1 + \sum_{n=1}^{M+1} \frac{b_n}{\mu_r^{2n}} \right) \\ g_{\text{cr}} = \frac{\mu_r^4}{4} \left(1 + \sum_{n=1}^{M+2} \frac{g_n}{\mu_r^{2n}} \right) \end{cases} \quad (5.7)$$

where we cut the expansion at order $1/\mu_r^{2M}$. For the polynomial $P(z)$ we assume

$$P(z) = 1 + \frac{1}{\mu_r^4} \sum_{i=1}^M \frac{e_i}{\mu_r^{2i}} + \frac{1}{\mu_r^8} \sum_{s=1}^{M-2} \frac{z^{2s}}{\mu_r^{2s}} \sum_{i=1}^{M-1-s} \frac{a_{s,i}}{\mu_r^{2i-2}}. \quad (5.8)$$

Note that, to $\mathcal{O}(1/\mu^{2M})$, the degree of $P(z)$ equals $2(M-2)$, fully consistent with (5.1).

To determine the coefficients b_n, g_n, e_i and $a_{s,i}$ we impose $\mathcal{G}[\mathcal{G}(z)] = z + \mathcal{O}(\mu^{-2M})$ and require the normalization of the density to be $1 + \mathcal{O}(\mu^{-2M})$. Practically, the normalization requirement amounts to asking that the coefficient of $1/z$ in the large z expansion of (5.6) equal one. The resulting system of equations can be solved recursively using the following two stage procedure. First, one expresses b_{M+1}, g_{M+2} and $a_{s,i}$ with $s+i = M-1$ in terms of e_M and the coefficients of lower order. Then e_M is determined by the normalization equation, but only at order $M+2$. In this manner, all the coefficients in (5.6) are determined unambiguously.

The $1/\mu$ expansion described above can be easily carried out to very high orders. Let us present here just the first few terms of this expansion,

$$\left\{ \begin{array}{l} \mu^2 = \mu_r^2 \left(1 - \frac{8}{\mu_r^4} - \frac{64}{\mu_r^8} + \frac{256}{\mu_r^{12}} - \frac{12800}{\mu_r^{16}} \right) + \mathcal{O}\left(\frac{1}{\mu_r^{18}}\right) \\ g_{\text{cr}} = \frac{\mu_r^4}{4} \left(1 - \frac{96}{\mu_r^8} + \frac{1024}{\mu_r^{12}} - \frac{30720}{\mu_r^{16}} + \frac{819200}{\mu_r^{20}} \right) + \mathcal{O}\left(\frac{1}{\mu_r^{20}}\right) \\ P(z) = 1 + \frac{2560}{\mu_r^{12}} - \frac{128z^2}{\mu_r^{10}} \left(5 + \frac{252}{\mu_r^4} \right) + \frac{5376z^4}{\mu_r^{12}} + \mathcal{O}\left(\frac{1}{\mu_r^{14}}\right) \end{array} \right. \quad (5.9)$$

The information about the critical properties of the theory is encoded in the value of $P(z=0)$. Indeed, the critical eigenvalue density $\rho(z)$ behaves at small z as $\pi\rho(z) \approx a(\mu^2)z^2$ with

$$\begin{aligned} a(\mu^2) &= \frac{\mu_r^3}{2\sqrt{2}} P(z=0) \\ &= \frac{\mu_r^3}{2\sqrt{2}} \left(1 + \frac{1}{\mu_r^4} \sum_{i=1}^M \frac{e_i}{\mu_r^{2i}} \right). \end{aligned} \quad (5.10)$$

If $a(\mu^2)$ is finite and nonzero, the critical index δ equals $\delta = 2$. On the other hand, a vanishing $a(\mu^2)$ would indicate a higher order multicritical point. Also, one could encounter a singularity where $a(\mu^2)$ goes to infinity. This would mean that for small x the density $\rho(x)$ vanishes slower than x^2 and the asymptotics $\rho(x) \propto x^2$ does not apply. In fact, we do expect such a singularity to occur at the Kosterlitz–Thouless point $\epsilon = 1$ or, equivalently, $\mu^2 = 2$. At that point [12]

$$\rho(x) \propto \frac{|x|}{\log[1/(\lambda|x|)]} \quad (5.11)$$

which is indeed more singular than $\rho(x) \propto x^2$.

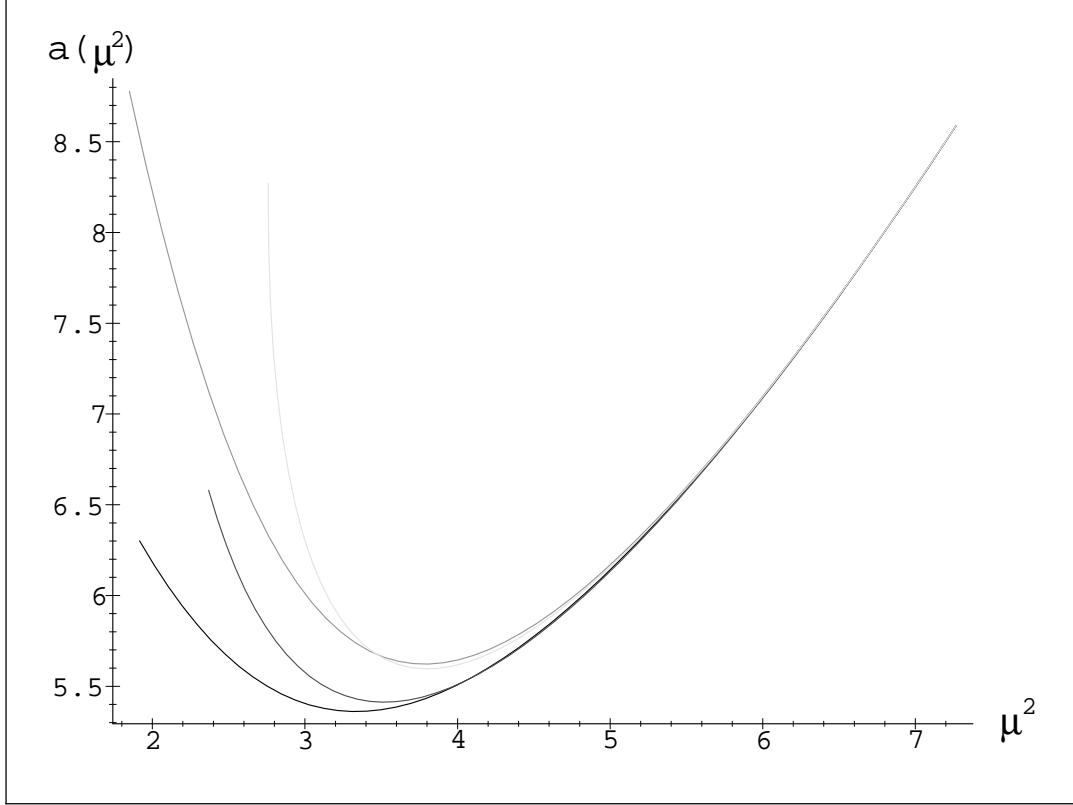


Fig. 5: The plots of $a(\mu^2)$ obtained from the strong coupling expansion with $M = 8, 10, 12$ and 14 (counting from lower left to upper right.) It is seen that $a(\mu^2)$ does not vanish and exhibits a singularity around $\mu^2 = 2$.

Therefore, the coefficient $a(\mu^2)$ should blow up at $\mu^2 = 2$. In fig. 5 we plot $a(\mu^2)$ computed from the strong coupling expansion up to $\mathcal{O}(\mu^{-28})$. These plots show that $a(\mu^2)$ certainly does not vanish. Therefore, no higher order multicriticalities occur for $\epsilon > 1$. Furthermore, $a(\mu^2)$ does blow up in the region around $\mu^2 = 2$. The Pade approximants reveal only one singularity in that region which we can therefore associate with the Kosterlitz–Thouless phase transition. Of course, to determine the critical value of μ^2 numerically is very hard, as the strong coupling expansion becomes unreliable when μ^2 gets less than ~ 3 . However, when combined with the theoretical results found in section 3, the numerical data become quite conclusive. Indeed, the universality arguments suggest that the difference between the “transcendental” potentials of section 3 and the pure quartic potential could only lead to a higher order multicritical point in the quartic case. But such points are reliably excluded by the data from the strong coupling expansion. At the same time, the exact position of the Kosterlitz–Thouless point can be derived from equation (2.25) independently of any numerical simulations.

To summarize, the strong coupling expansion, taken together with the exact results, indicates that the infinite chain model with a pure quartic potential exhibits the critical index $\delta = 2$ for any $\epsilon > 1$.

6. Conclusions

In this paper we have explored the $\epsilon > 1$ regime of the infinite random matrix chain. We have calculated the critical exponents and found that, for $\epsilon > 1$, they are the same as the critical exponents in one-matrix models. As a consequence, the critical behavior of the strongly coupled two-dimensional $O(2)$ model on a random planar lattice turns out to coincide with the critical behavior of pure two-dimensional gravity.

From the technical viewpoint, we found that most properties of the chain model are encoded in the midway density $\rho_{1/2}(x)$ which enters the picture through the hydrodynamic representation. We have seen that the changes in the branch structure of $\rho_{1/2}$ govern the Kosterlitz–Thouless phase transition and explain the change in critical exponents.

Finally, we constructed the character expansion for the matrix chain. Rather remarkably, in the character expansion language the mechanisms of the Kosterlitz–Thouless phase transition and the Douglas–Kazakov-type transitions are precisely the same. We derived an exact relation between the “Young tableau density” of the representation dominating the character sum at large N and the hydrodynamic density $\rho_{1/2}$.

There are, however, a number of problems that remained open. First of all, it would be desirable to engineer an exact solution reproducing the critical behavior of the matrix chain on both sides of $\epsilon = 1$. According to (5.11), the requisite midway density $\rho_{1/2}$ would have to contain a logarithmic singularity and, for $\epsilon > 1$, have the two-branch analytic structure. We are confident that such a solution exists and can be constructed with our methods.

Furthermore, it would be very interesting to interpret our results in terms of another theory describing the two-dimensional $O(2)$ model—the matrix quantum mechanics on a circle. There, too, the representation structure changes at the Kosterlitz–Thouless point, although in a somewhat different way. Since the chain model and the compactified matrix quantum mechanics are dual to each other, there might exist a correspondence between the large N limits of their character expansions.

In this respect, the matrix theory on a circle resembles the instanton description of QCD_2 . Indeed, two-dimensional QCD on a sphere has two dual descriptions. One of them is the character expansion given by the Migdal–Rusakov formula [13][21]. The other represents the partition function of QCD_2 as a sum over all instanton sectors, and can be obtained from the character expansion by Poisson resummation [15].

At large N the representation dominating in the character expansion formula has $\mathcal{O}(N)$ boxes and, depending on the phase, may or may not have a gap. The Young tableaux corresponding to such representations are shown in fig. 1. On the other hand, for $N \rightarrow \infty$ the instanton expansion is also dominated by a single sector. The N charges parametrizing the dominant instanton all strictly equal zero at weak coupling but become nonzero integers of order N at strong coupling [15][19].

This is very similar to what happens in the matrix quantum mechanics on a circle. There the vortexless phase is dominated by the trivial (singlet) representation, with zero boxes in each Young tableau row [5][6]. In the other phase, where the vortices are significant, the dominant representation has $\mathcal{O}(N)$ boxes. We see that the row lengths of the dominant Young tableau in the matrix quantum mechanics behave the same way as the instanton charges in QCD_2 .

All of this suggests that, by analogy with QCD_2 , a simple Poisson resummation of the character expansion formula (4.9) could tell us a lot about compactified matrix quantum mechanics. If true, such analysis could be extremely useful, as the matrix quantum mechanics on a circle so far appeared intractable by any known computational method.

It is our pleasure to thank D. Gross, V. Kazakov and A.A. Migdal for their interest, support and for helpful advice.

This work is supported in part by funds provided by the U.S. Department of Energy under cooperative research agreement DE-FC02-94ER40818 and the Swiss National Science Foundation.

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